Quasiconvex problems and Parabolic systems

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Minicourse, Mathematical Institute
Oxford, October 2015

Quasiconvex problems

What about Quasiconvex functionals?
A function $f : \mathbb{R}^{nN} \to \mathbb{R}$ is quasiconvex iff

$$\int_Q [f(A + D\eta) - F(A)]dx \geq 0$$

for every $A \in \mathbb{R}^{nN}$ and every $\eta \in C_0^1(Q, \mathbb{R}^N)$. The original definition is due to Morrey in the 50's.
He proved a lower semicontinuity result with respect to weak* topology in $W^{1,\infty}$.

For physical motivations related to models of nonlinear elasticity, see Ball, Arch.77
Quasiconvexity and growth condition guarantee existence of minimizers in Sobolev space:(Acerbi& Fusco Arch. 84)

$$0 \leq f(A) \leq K(1 + |A|^p)$$

then quasiconvexity $\Leftrightarrow$ w.l.s.c.i in $W^{1,p}$.
convexity $\Longrightarrow$polyconvexity $\Longrightarrow$ quasiconvexity
Quasiconvexity is a non-local condition, Kristensen Ann.Poincaré 99.

Evans considered uniform strict quasiconvexity

$$\int_{B_1} f(A + D\eta) - f(A) \geq \nu \int_{B_1} (\mu + |A| + |D\eta|)^{p-2}|D\eta|^2$$

for every $A \in \mathbb{R}^{nN}$ and every $\eta \in C^\infty(B_1, \mathbb{R}^N)$.
and proved partial regularity for quasiconvex functionals .

Assumptions

- $f \in C^1(\mathbb{R}^{nN}) \bigcap C^2(\mathbb{R}^{nN} \setminus 0)$;
- $|f(Q)| \leq K|Q|^p$;
- $|(D^2 f)(Q)| \leq |Q|^{p-2}$;
- Hölder continuity of $D^2 f$ away from zero.

$$\int_{B_1} f(A + D\eta) - f(A) \geq \nu \int_{B_1} (\mu + |A|^2 + |D\eta|^2)|^{\frac{p-2}{2}} |D\eta|^2$$

for every $A \in \mathbb{R}^{nN}$ and every $\eta \in C_0^\infty(B_1, \mathbb{R}^N)$ with compact support.

- $\mu > 0$ Non degenerate case
- $\mu = 0$ Degenerate case

Idea for partial regularity

- $\mathcal{A}$-harmonic approximation $(\mathcal{A} = D^2 f(Q))$
- interpolation inequality
- excess decay estimate for the harmonic map h

$$\Downarrow$$

- $C^{1,\alpha}$-partial regularity.

Main Theorem (Diening, Lengeler, S., Verde)

Let u be a local minimizer of the uniform strict quasiconvex functional.
Denote by $\mathcal{R}(u)$ the set of regular points of u. Then

$$\mathcal{R}(u) = \{x_o \in \Omega : \liminf_{r\to 0}\Phi(B(x_0, r), u) = 0\}$$

Moreover, if $x_o \in \mathcal{R}(u)$ and

$$\limsup_{r\to 0} \frac{|(V(Du))_{x_0,r}|^2}{\Phi(B(x_0, r), u)} = +\infty \tag{1}$$

then there exists a $\sigma > 0$ such that $Du \in C^{0,\beta}(B_\sigma(x_0))$ for any $\beta \in (0, 1)$. In particular, this holds for the regular points x_0 such that $Du(x_0) \neq 0$.

Consider a bilinear form on *Hom*$(\mathbb{R}^{nN}, \mathbb{R}^{nN})$ which is (strongly) elliptic in the sense of Legendre-Hadamard, i.e. if for all $a \in \mathbb{R}^N, b \in \mathbb{R}^n$ it holds

$$\mathcal{A}_{ij}^{\alpha\beta} a^i b_\alpha a^j b_\beta \geq \kappa_A |a|^2 |b|^2$$

for some $\kappa_A > 0$.
The function u is *almost $\mathcal{A}$-harmonic*, iff:

$$\left| \fint_B \mathcal{A} \nabla u \cdot \nabla \xi \, dx \right| \leq \delta \fint_{\widetilde{B}} |\nabla u| \, dx \, \|\nabla \xi\|_{L^\infty(B)} \tag{2}$$

for all $\xi \in C_0^\infty(B, \mathbb{R}^N)$.

Given a Sobolev function u on a ball B we want to find an $\mathcal{A}$-harmonic function h which is close to our function u. The function h will be the $\mathcal{A}$-harmonic estension of u on ∂B:

$$\begin{aligned} -div(\mathcal{A}\nabla h) &= 0 && \text{on } B \\ h &= u && \text{on } \partial B \end{aligned} \tag{3}$$

in the sense of distributions.
Let $w := h - u$, then we can find, equivalently, a Sobolev function w which satisfies

$$\begin{aligned} -div(\mathcal{A}\nabla w) &= -div(\mathcal{A}\nabla u) && \text{on } B \\ w &= 0 && \text{on } \partial B \end{aligned} \tag{4}$$

in the sense of distributions.

For every $G \in L^q(B, \mathbb{R}^{nN})$ there exists a unique $u = T_{\mathcal{A}}(G) \in W_0^{1,q}(B, \mathbb{R}^N)$ solution of

$$\begin{aligned} -div(\mathcal{A}\nabla u) &= -div(G) && \text{on } B \\ u &= 0 && \text{on } \partial B \end{aligned} \tag{5}$$

The solution operator $T_{\mathcal{A}}$ is linear and satisfies

$$\|\nabla T_{\mathcal{A}}(G)\|_q \leq c\|G\|_q$$

In addition, $T_{\mathcal{A}} : L^q \to W_0^{1,q}$ for every $q \in (1, \infty)$. One can use interpolation and extend to convex function ψ:

$$\int_B \psi(|\nabla T_{\mathcal{A}}(G)|)dx \leq c \int_B \psi(|G|)dx$$

Duality lemma

Let $B \subset \Omega$ be a ball and let $\mathcal{A}$ be strongly elliptic in the sense of Legendre-Hadamard. Then it holds for all $u \in W_0^{1,\psi}(B)$ that

$$\|\nabla u\|_{\psi} \sim \sup_{\substack{\xi \in C_0^\infty(B) \\ \|\nabla \xi\|_{\psi^*} \leq 1}} \int_B \mathcal{A} \nabla u \cdot \nabla \xi \, dx,$$

$$\int_B \psi(|\nabla u|)\, dx \sim \sup_{\xi \in C_0^\infty(B)} \left[\int_B \mathcal{A} \nabla u \cdot \nabla \xi \, dx - \int_B \psi^*(|\nabla \xi|)\, dx \right].$$

The implicit constants only depend on n, N, κ_A, $|\mathcal{A}|$, $\Delta_2(\psi, \psi^*)$.

$\mathcal{A}$-harmonic approximation in Orlicz spaces

Let ψ be an N-function and let $s > 1$. Then for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \psi, s) > 0$ such that the following holds: let $u \in W^{1,\psi}(\widetilde{B})$ be *almost $\mathcal{A}$-harmonic* on B. Then the unique solution $w = h - u \in W_0^{1,\psi}(B)$ satisfies

$$\fint_B \psi\Big(\frac{|w|}{r_B}\Big)\,dx + \fint_B \psi(|\nabla w|)\,dx$$
$$\leq \varepsilon\bigg(\Big(\fint_B (\psi(|\nabla u|))^s\,dx\Big)^{\frac{1}{s}} + \fint_{\widetilde{B}} \psi(|\nabla u|)\,dx\bigg).$$

The proof applies the Lipschitz truncation to the test function ξ and uses the duality Lemma.

Parabolic Lipschitz truncation: (Diening, Schwarzacher, S., Verde)

- The idea is to regularize a given function by cutting off regions of irregularity and then to extend this restricted function to the whole domain again by a Whitney type covering argument
  introduced firstly by Kinnunen-Lewis to prove higher integrability of very weak solutions and next by Diening-Ruzicka-Wolf in the context of Newtonian fluids .

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- MAIN DIFFERENCES w.r.t. the previous approaches: Our truncation preserve boundary values. In addition, we get the Poincaré inequality as a consequence and not as additional hypothesis.

- Decompose the bad set O_λ^α into a family of cylinders (Q_j^α) (Whitney decompostion); $(Q_r^\alpha = (-\alpha r^2, \alpha r^2) \times B_r)$
- Let $\phi_j^\alpha \in C_0^\infty(R^{m+1})$ be a partition of unity w.r. t Q_j^α, such that $\chi_{\frac{1}{2}Q_j^\alpha} \leq \phi_j^\alpha \leq \chi_{\frac{3}{4}Q_i^\alpha}$

$$w_j^\alpha := \begin{cases} \fint_{Q_j^\alpha} w\phi_j^\alpha & \text{if } \frac{3}{4}Q_j^\alpha \subset J \times \Omega, \\ 0 & \text{else.} \end{cases}$$

We define our truncation w_λ^α via the formula

$$w_\lambda^\alpha := w - \sum_j \phi_j^\alpha (w - w_j^\alpha).$$

Since the ϕ_j^α are locally finite, the sum is pointwise well defined.

$$\begin{aligned} \partial_t w &= divG \qquad \text{in } \mathcal{D}'(R \times \Omega) \\ w &= 0 \qquad \text{outside of } J \times \Omega, \\ G &= 0 \qquad \text{outside of } J \times \Omega. \end{aligned}$$

For $\alpha, \lambda > 0$ consider the set

$$O_\lambda^\alpha := \{M_\alpha(\nabla w) > \lambda\} \cup \{\alpha M_\alpha(G) > \lambda\}$$

We use the following α-parabolic maximal function

$$M_\alpha(\nabla w)(x) := \sup_{\{Q \in \mathcal{Q}^\alpha \,:\, x \in Q\}} \mathcal{M}_Q(\nabla w).$$

$$Q_r^\alpha = (-\alpha r^2, \alpha r^2) \times B_r.$$

Estimates on Whitney cubes

Poincaré inequality

There holds

$$\fint_{\frac{3}{4}Q_j^\alpha} \frac{|w - w_j^\alpha|}{r_j}\, dz + \fint_{Q_j^\alpha} |\nabla w| dz + \alpha \fint_{Q_j^\alpha} |G| dz \leq c\,\lambda.$$

Lipschitz property

$$\sum_{j \in A_k} \frac{|w_j^\alpha - w_k^\alpha|}{r_j} \leq c \sum_{j \in A_k} \fint_{\frac{3}{4}Q_j^\alpha} \frac{|w - w_j^\alpha|}{r_j}\, dz \leq c\,\lambda.$$

Estimates for w_λ^α

Spatial gradient

For $\alpha = \lambda^{2-p}$. If $\nabla w \in L^p(J \times \Omega)$ and $\partial_t w = divG$ with $G \in L^{p'}(J \times \Omega)$, then $\nabla w_\lambda^\alpha \in L^p(J \times \Omega)$.Moreover,

$$\int_{O_\lambda^\alpha} |\nabla(w - w_\lambda^\alpha)| dxdt \leq c \int_{O_\lambda^\alpha} (|\nabla w|^p + |G|^{p'}) dxdt$$

Time derivative

If $\nabla w \in L^p(J \times \Omega)$ and $\partial_t w = divG$ with $G \in L^{p'}(J \times \Omega)$, then for every ξ with $\nabla \xi \in L^{p'}, \partial_t \xi = divH_\xi$ and $H_\xi \in L^p(J \times \Omega)$, we find

$$|\langle \partial_t (w - w_\lambda), \xi \rangle| \leq c \int_{O_\lambda^\alpha} |\nabla w|^p + |G|^{p'} + |H_\xi|^{p'} + |\nabla \xi|^p \, dx \, dt.$$

Hölder continuity

Corollary

w_λ^α is Hölder continuous with respect to d^α, i.e.

$$|w_\lambda^\alpha(t,x) - w_\lambda^\alpha(s,y)| \le c\,\lambda \max\left[\frac{|t-s|^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}}, |x-y|\right]$$

Good λ inequality

Define

$$\gamma^p = \fint_Q |\nabla w|^p dz + \fint_Q |G|^{p'} dz$$

For every $m_0 \in \mathbb{N}$ there exists a $\lambda \in [\gamma, 2^{m_0}\gamma]$ such that for $\alpha(\lambda) = \lambda^{2-p}$ we have:

$$|\{M_\alpha(\nabla w) > \lambda\}| + |\{M_\alpha(G) > \lambda^{p-1}\} \leq c \frac{\gamma^p}{m_0 \lambda^p} |Q|$$

Idea of DiBenedetto's intrinsic geometry approach.

The presence in the parabolic p-Laplacian system

$$h_t = div(|\nabla h|^{p-2}\nabla u)$$

of both an evolutionary and a p-growth term: h_t and $|\nabla h|^{p-2}\nabla h$

$$\Downarrow$$

lack of scaling that rules out the possibility of reverse-type inequalities since they are homogeneous in nature

On the other hand, the resulting lack of homogeneity of local estimates does not allow iterations like in the elliptic case and must be re-balanced in some way.

Outline: how such the classical intrinsic approach works!

Let us consider a cylinder Q, where, roughly speaking, the size of the gradient is approximately λ

$$\Downarrow$$

In this case we shall consider cylinders of the type

$$Q_\varrho^\lambda = Q_\varrho^\lambda(z_0) = (t_0 - \lambda^{2-p}\varrho^2, t_0) \times B_\varrho(x_0).$$

We have that the p-parabolic system looks like

$$h_t = div(\lambda^{p-2}\nabla h)$$

which after a scaling, that is considering

$$(x,t) \in (-1,0) \times B_1(0) \to v(x,t) := h(x_0 + \varrho x, t_0 + \lambda^{p-2}\varrho^2 t),$$

behaves exactly as the heat system

$$v_t = \Delta v \quad (-1,0) \times B_1(0),$$

Intrinsic geometry

Thus the success of this strategy is linked to the construction of such cylinders. Indeed, the right way is to consider

$$\Big(\frac{1}{|Q_\varrho^\lambda|}\int_{Q_\varrho^\lambda} |\nabla h|^p dz\Big)^{\frac{1}{p}} \approx \lambda.$$

i.e. the value of the integral average must be comparable to a constant which is in turn involved in the construction of Q_ϱ^λ.

Estimates for h p-caloric

If $Q_\varrho^\lambda \subset Q_T$ is K-subintrinsic $\left(\frac{1}{|Q_\varrho^\lambda|}\int_{Q_\varrho^\lambda} |\nabla h|^p dz \leq K\lambda^p\right)$, then it holds:

$$\sup_{\frac{1}{2}Q_\varrho^\lambda} |\nabla h| \leq c\lambda$$

Theorem

If

$$\left(\frac{1}{|Q_\varrho^\lambda|}\int_{Q_\varrho^\lambda} |\nabla h|^p dz\right)^{\frac{1}{p}} \approx \lambda$$

then there exist $C > 0, \alpha, \tau \in (0,1)$ *such that* $\forall \theta \in (0,\tau]$

$$\sup_{z,w \in \theta Q_\varrho^\lambda} |V(\nabla h(w)) - V(\nabla h(z))|^2$$

$$\leq c\theta^\alpha \fint_{Q_\varrho^\lambda} |V(\nabla h) - (V(\nabla h))_{Q_\varrho^\lambda}|^2 dx$$

p-caloric approximation Lemma

We show that every “almost p-caloric” function has a p-caloric approximation “close enough”, roughly speaking:

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Let $\sigma \in (0,1)$ and $\theta \in (0,\frac{1}{2})$. Then $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. the following holds: If u is almost p-caloric in the sense that $\forall \xi \in C_0^\infty$,

$$\left| \fint_Q u\partial_t\xi + |\nabla u|^{p-2}\nabla u \nabla \xi \, dz \right| \le \delta\Big[\fint_Q |\nabla u|^p + |H|^{p'} dz + \|\nabla\xi\|_\infty^p\Big]$$

and $\partial_t u = divH$, then there exists a p-caloric function h s.t. $h = u$ on $\partial_p Q$

$$\fint_I \Big(\fint_B \frac{|u-h|^{2\sigma}}{|t^+ - t^-|} dx\Big)^{\frac{1}{\sigma}} + \Big(\fint_Q |\nabla u - \nabla h|^{p\theta} dz\Big)^{\frac{1}{\theta}} \le \varepsilon \fint_Q |\nabla u|^p + |H|^{p'} dz$$

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- The proof is achieved by using Lipschitz truncation argument.
- MAIN DIFFERENCES w.r.t. the previous approaches: direct proof (not by contradiction), $h = u$ on the boundary and in particular a quantitative estimate in terms of the closeness of gradients.

Sketch of the p-caloric approximation: We have $\partial_t u = divH$

- Take h solution of the homogeneous problem in a cylinder Q with $h = u$ on ∂Q(local p-caloric comparison function). Observe that $w = h - u$ satisfies:

$$\partial_t w = div(H - |\nabla u|^{p-2}\nabla u) = divG$$

- Let $\gamma > 0$ s.t. $\gamma^p = ⨍_Q |\nabla u|^p \, dz + ⨍_Q |H|^{p'}$ and we select $\lambda \in [\gamma, 2^{m_0}\gamma]$ (m_0 will be very large). and w_λ Lipschitz truncation. We consider $\xi = w_\lambda \eta$ as test function in both problems ($\eta = \frac{t^+ - t}{t^+ - t^-}$) (*almost* p-caloric estimate and p-caloric system);
- Monotonicity of the operator, Young's inequality ,good estimates for the comparison function h and useful properties of w_λ.

Parabolic Systems with critical growth, $p > \frac{2m}{m+2}$

$$\partial_t u - div(|Du|^{p-2}Du) = F(Du)$$

$$|F(P)| \leq c|Du|^p$$

$$\|u\|_\infty \text{small}$$

Higher integrability of Du and $\frac{1}{|B_R|}\int_{Q_R}|Du|^p dz < \varepsilon$

$$\Downarrow$$

$$\fint_{Q_r}|Du|^p dz \leq cr^{-\alpha}$$

(Morrey estimate) $\forall r \leq 2R$, $\forall \alpha > 0$

$$\Downarrow$$

u Hölder continuous

How to prove the result?

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Main ingredients:

- DiBenedetto's intrinsic geometry approach
- Discrete estimates on subintrinsic cylinders via *p-caloric approximation*

We will prove first a discrete Morrey estimate, valid on rescaled cylinders.

Lemma

Let u be a bounded weak solution satisfying smallness in norm and fix $\delta > 0$. Then if we assume for some $z_0 \in Q$ and $R_0 > 0$ that it holds

$$\frac{1}{R_0^m} \int_{Q((4R_0)^p, 4R_0)(z_0)} |Du|^p \, dz \leq \delta^q \tag{7}$$

then

$$\int_{Q(\lambda_k^{2-p} \rho_k^2, \rho_k)(z_0)} |Du|^p \, dz \leq \lambda_k^p \tag{8}$$

for $\lambda_k = M\lambda_0, \rho_k = \tau \rho_{k-1}, M > 1, \tau < 1$.

Discrete estimates: Using the p- caloric approximation, we prove iterative local energy estimates on subintrinsic cubes.

- 1. Step: $\exists \lambda_k, \varrho_k$ s.t.

$$\fint_{Q(\lambda_k^{2-p}\varrho_k^2, \varrho_k)} |Du|^p \leq \lambda_k^p \tag{9}$$

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Hint: We choose suitable λ_0, ϱ_0 s. t. (9) holds and we proceed by induction.
Main tool: comparison with a solution of the homogeneous problem coinciding with u on the parabolic boundary.

Assuming

$$\fint_{Q(\lambda_{k-1}^{2-p}\rho_{k-1}^{2},\rho_{k-1})} |Du|^p \, dz \leq \lambda_{k-1}^{p} \tag{10}$$

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$$\Downarrow$$

Approximate *p*- caloricity

$$\left|\fint_{Q_{\rho_k}^{\lambda_k}} u\varphi_t - |Du|^{p-2}Du \cdot D\varphi \, dz\right|$$

$$\leq c\delta^s \left[\|D\varphi\|_\infty^p + \fint_{Q_{\rho_k}^{\lambda_k}} |Du|^p\right].$$

The p-caloric approximation Lemma and the higher integrability result imply that there exists a p-caloric function h such that

$$\fint_{Q_{\rho_k}^{\lambda_k}} |Du - Dh|^p \leq \varepsilon \cdot \left(\fint_{Q_{\rho_{k-1}}^{\lambda_{k-1}}} |Du|^{p+\varepsilon_0} \right)^{\frac{p}{p+\varepsilon_0}}.$$

We get

$$\fint_{Q_{\rho_k}^{\lambda_k}} |Du|^p \leq \lambda_k^p$$

By induction the cylinders are subintrinsic.

- 2. Step: By using covering argument, we adapt the results of Leone-Misawa-Verde to get

$$\int_{Q_{r_k^p, r_k}} |Du|^p \leq c r_k^{m+p-\delta}, \quad \forall \delta < p$$

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- 3. Step Conclusion: $\forall r_{k+1} < r \leq r_k$ we get

$$\int_{Q_{r^p, r}} |Du|^p \leq c r^{m+p-\delta}, \quad \forall \delta < p$$

Summarizing:

For the parabolic system with critical growth we get

small energy of u

$$\Downarrow$$

Higher integrability of Du and $\frac{1}{|B_R|}\int_{Q_R}|Du|^p dz < \varepsilon$

$$\Downarrow$$

Hölder continuity of u with respect to the metric $|t|^{\frac{1}{p}} + |x|$.

Using again a covering argument, we get

partial Hölder continuity

of ∇u with respect to the parabolic metric $|t|^{\frac{1}{2}} + |x|$.